§ 4 Primary decomposition

UFD $\rightsquigarrow$ $x = u \cdot \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r}$

$\mathbb{Z}[\sqrt{-5}] \neq$ UFD $\qquad 6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

怎么推广?

prime $\rightsquigarrow$ prime ideal

power of prime $\rightsquigarrow$ <u>primary ideal</u>

<u>Def</u>: An ideal $q \triangleleft A$ is primary if $q \neq A$ and if

$xy \in q \Rightarrow x \in q$ or $y \in \sqrt{q}$

<u>Fact</u>: 1) $q \triangleleft A$ primary $\Leftrightarrow$ $A/q \neq 0$ and every zero divisors in A/q is nilpotent.

2) prime ideal is primary

3) contraction of a primary ideal is primary.

Pf: $f: \underset{\substack{\cup \\ q^c}}{A} \to \underset{\substack{\triangledown \\ q}}{B} \Rightarrow A/q^c \hookrightarrow B/q$

$\overset{1)}{\Longrightarrow} \checkmark$

<u>Prop 4.1</u> q = primary $\Rightarrow$ $\sqrt{q}$ = the smallest prime ideal containing q.

Pf: $\sqrt{\mathfrak{q}} = \bigcap_{\mathfrak{p} \supset \mathfrak{q} : \text{prime}} \mathfrak{p} \Rightarrow$ ONTS: $\sqrt{\mathfrak{q}}$ = prime.

$\forall\ xy \in \sqrt{\mathfrak{q}} \Rightarrow x^n y^n \in \mathfrak{q}$

$\Rightarrow x^n \in \mathfrak{q}$ or $y^{nm} \in \mathfrak{q}$

$\Rightarrow x \in \sqrt{\mathfrak{q}}$ or $y \in \sqrt{\mathfrak{q}}$ $\square$

Def: A primary $\mathfrak{q}$ is called $\mathfrak{p}$-primary, if $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

Example: 1) primary ideal in $\mathbb{Z}$. (0), (p^n).

2) primary ideal is not necessarily a prime-power.

$\mathfrak{q} = (x, y^2) \triangleleft A = k[x,y]$. $\mathfrak{p} = \sqrt{\mathfrak{q}} = (x,y)$, $\mathfrak{p}^2 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$.

- $A/\mathfrak{q} \cong k[y]/y^2$. zero divisors = nilpotents.

3) prime power is not necessarily primary.

$\mathfrak{p} = (\bar{x}, \bar{z}) \triangleleft A = k[x,y,z]/(xy - z^2)$

$\mathfrak{q} = \mathfrak{p}^2$. $\bar{x}\bar{y} = \bar{z}^2 \in \mathfrak{q}$. but $\bar{x} \notin \mathfrak{q}$ & $\bar{y} \notin \mathfrak{p} = \sqrt{\mathfrak{q}}$.

Prop 4.2: $\sqrt{\mathfrak{a}}$ = maximal $\Rightarrow \mathfrak{a}$ = primary.
in particular, $\mathfrak{m}$ = maximal $\Rightarrow \mathfrak{m}^n$ = $\mathfrak{m}$-primary.

Pf: $\mathfrak{m} := \sqrt{\mathfrak{a}}$ = max $\Rightarrow \mathfrak{m}/\mathfrak{a} \triangleleft A/\mathfrak{a}$ the only one prime ideal

$\Rightarrow$ either unit or nilpotent

$\Rightarrow$ zero divisor is nilpotent. $\square$

Lemma 4.3. $q_i = \mathfrak{p}$-primary $(1 \le i \le n) \Rightarrow q := \bigcap_{i=1}^{n} q_i = \mathfrak{p}$-primary.

Pf: $\cdot$ $\sqrt{q} = \sqrt{\cap q_i} = \cap \sqrt{q_i} = \mathfrak{p}$

$\cdot$ $\forall\, xy \in q,\ x \notin q \Rightarrow \exists i \quad xy \in q_i,\ x \notin q_i$

$$\Rightarrow y \in \sqrt{q_i} = \mathfrak{p}$$

□

Lemma 4.4. $q = \mathfrak{p}$-primary. $x \in A$. Then

1) $x \in q \Rightarrow (q:x) = A$

2) $x \notin q \Rightarrow (q:x) = \mathfrak{p}$-primary $\left(\Rightarrow \begin{cases} q \subseteq (q:x) \subseteq \mathfrak{p} \\ \sqrt{(q:x)} = \mathfrak{p} \end{cases} \right)$

3) $x \notin \mathfrak{p} \Rightarrow (q:x) = q$

Pf: 1) & 3) by definition.

2): $x \notin q \Rightarrow q \subseteq (q:x) \subseteq \mathfrak{p} \Rightarrow \sqrt{(q:x)} = \mathfrak{p}$

$$\left.\begin{aligned} \forall\, \alpha\beta &\in (q:x) \\ \alpha &\notin (q:x) \end{aligned}\right\} \Rightarrow \begin{cases} \alpha\beta x \in q \\ \alpha x \notin q \end{cases}$$

$$\Rightarrow \beta \in \sqrt{q} = \mathfrak{p} = \sqrt{(q:x)}$$

Def A *primary decomposition* of $\mathfrak{a} \triangleleft A$ is an expression of $\mathfrak{a}$ as a finite intersection of primary ideals

$$\mathfrak{a} = \bigcap_{i=1}^{n} q_i .$$

- minimal, if
  - $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j} \quad \forall i, j$
  - $\bigcap_{j \neq i} \mathfrak{q}_j \not\subseteq \mathfrak{q}_i$

Fact: any primary decomposition can be reduced to a minimal one.

<u>Def</u>: • $\mathfrak{a}$ is <u>decomposable</u>, if $\exists$ primary decomp.

<u>Thm</u> 4.5 (1st uniqueness theorem) $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal primary decomp.

$$\{ \sqrt{\mathfrak{q}_i} \mid i \} = \{ \sqrt{(\mathfrak{a}:x)} \mid x \in A \} \cap \{ \text{prime ideals} \}$$

Pf: • $(\mathfrak{a}:x) = (\bigcap \mathfrak{q}_i : x) = \bigcap (\mathfrak{q}_i : x)$

$$\Rightarrow \sqrt{(\mathfrak{a}:x)} = \sqrt{\bigcap_i (\mathfrak{q}_i : x)} = \bigcap_i \sqrt{(\mathfrak{q}_i : x)} \overset{4.4}{=} \bigcap_{x \notin \mathfrak{q}_i} \sqrt{\mathfrak{q}_i}$$

• "$\supseteq$": Suppose $\sqrt{(\mathfrak{a}:x)}$ = prime $\Rightarrow \sqrt{(\mathfrak{a}:x)} = \sqrt{\mathfrak{q}_i}$ for some i.

• "$\subseteq$": minimal $\Rightarrow \forall i \; \exists x_i \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$

$$\Rightarrow \sqrt{(\mathfrak{a}:x_i)} = \mathfrak{q}_i$$

Rmk: i) $\{\sqrt{\mathfrak{q}_i} \mid i\}$ does not depend on the choice of decomp.

ii) $\forall i\ \exists x_i$ s.t. $(\mathfrak{a} : x_i)$ is $\mathfrak{p}_i$-primary.

Example: $\sqrt{\mathfrak{a}} = \text{prime} \not\Rightarrow \mathfrak{a} = \text{primary}$

$\cdot\ \mathfrak{a} = (x^2, xy)\left(\triangleleft A[x,y]\right) = (x) \cap (x,y)^2$

$\cdot\ \sqrt{\mathfrak{a}} = (x)$

Let $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ be a m.p.d.. $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$

$\Sigma = \{\mathfrak{p}_i \mid i\} \supseteq \{\mathfrak{p}_i \mid \underbrace{\mathfrak{p}_i \not\supseteq \mathfrak{p}_j\ \forall j \neq i}_{\text{i.e. } \mathfrak{p}_i \text{ minimal}}\} = \Sigma_{min}$

Σ prime ideals belong to $\mathfrak{a}$.

$\shortparallel$

Σ_{min} minimal (isolated) prime ideals belong to $\mathfrak{a}$.

$\amalg$

$(\Sigma \setminus \Sigma_{min})$ embedding prime ideals belong to $\mathfrak{a}$.

Prop 4.6 $\mathfrak{a}, \mathfrak{q}_i, \mathfrak{p}_i$ as above. Then

$$\mathfrak{p} \supseteq \mathfrak{a} \text{ prime} \Rightarrow \exists\, \mathfrak{p}_i \text{ s.t. } \mathfrak{p}_i \subseteq \mathfrak{p}.$$

Pf: $\mathfrak{p} \supseteq \mathfrak{a} = \bigcap \mathfrak{q}_i \Rightarrow \mathfrak{p} \supseteq \sqrt{\mathfrak{a}} = \bigcap \sqrt{\mathfrak{q}_i} = \bigcap_i \mathfrak{p}_i$

$\Rightarrow \exists\, i$ s.t. $\mathfrak{p} \supseteq \mathfrak{p}_i$ $\square$

Cor: $\Sigma_{\min} = \left\{ \begin{array}{l} \text{minimal elements in the set of} \\ \text{all prime ideals containing } \mathfrak{a} \end{array} \right\}$

Prop 4.7 $\mathfrak{a} = \bigcap \mathfrak{q}_i$ minimal, $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. Then

$$\bigcup_{i=1}^{n} \mathfrak{p}_i \overset{\text{"}\supseteq\text{" easy part}}{=} \left\{ x \in A \;\middle|\; (\mathfrak{a} : x) \neq \mathfrak{a} \right\}$$

i.e. $(\mathfrak{a} : x) = \mathfrak{a} \Leftrightarrow x \notin \mathfrak{p}_j \ \forall j$.

Pf: Case $\mathfrak{a} = 0$: $\text{RHS} = D \overset{1.15}{=} \bigcup_{x \neq 0} \sqrt{(0:x)}$

$\sqrt{(0:x)} = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j \subseteq \text{LHS} \Rightarrow \text{RHS} \subseteq \text{LHS}$

$\forall i\ \exists\, x$ s.t. $\mathfrak{p}_i = \sqrt{(0:x)} \subseteq \text{RHS} \Rightarrow \text{LHS} \subseteq \text{RHS}$

$\bar{0} = \bigcap_i (\mathfrak{q}_i / \mathfrak{a})$ in $A/\mathfrak{a}$ & $\mathfrak{q}_i/\mathfrak{a}$ = primary in $A/\mathfrak{a}$

$\Rightarrow \bigcup_{i=1}^{n} (\mathfrak{p}_i / \mathfrak{a}) = \left\{ \bar{x} \in A/\mathfrak{a} \;\middle|\; (\bar{0} : \bar{x}) \neq \bar{0} \right\} \Rightarrow \checkmark$ $\square$

Fact. $0 = \bigcap \mathfrak{q}_i$ minimal. Then

i). $D :=$ set of zero divisors $= \bigcup_i \sqrt{\mathfrak{q}_i}$

ii). $\sqrt{0} :=$ set of nilpotent elements $= \bigcap_i \sqrt{\mathfrak{q}_i}$

Prop 4.8 (localization of primary ideal) $\mathfrak{q} = \mathfrak{p}$-primary.

$S =$ mult. closed subset. Then

i): $S \cap \mathfrak{p} \neq \phi \Rightarrow S^{-1}\mathfrak{q} = S^{-1}A$

ii): $S \cap \mathfrak{p} = \phi \Rightarrow S^{-1}\mathfrak{q} = S^{-1}\mathfrak{p}$-prime & $(S^{-1}\mathfrak{q})^c = \mathfrak{q}$

$$
\begin{array}{ccc}
\{\text{prime} \cdots\} & \xleftrightarrow{1:1} & \{\text{prime}\} \\
\cap & & \cap \\
\{\text{contracted primary ideals in } A\} & \xleftrightarrow{1:1} & \{\text{primary ideals in } S^{-1}A\} \\
\cap & & \cap \\
\{\text{contracted ideal in } A\} & \xleftrightarrow{1:1} & \{\text{ideals in } S^{-1}A\}
\end{array}
$$

Pf: i) $s \in S \cap \mathfrak{p} \Rightarrow s^n \in S \cap \mathfrak{q} \neq \phi$

$\Rightarrow S^{-1}\mathfrak{q} = S^{-1}A$

(ii) $S \cap \mathfrak{q} = \phi \overset{3.11}{\Rightarrow} \mathfrak{q}^{ec} = \mathfrak{q}$.

- $\sqrt{\mathfrak{q}^e} = \sqrt{S^{-1}\mathfrak{q}} = S^{-1}\sqrt{\mathfrak{q}} = S^{-1}\mathfrak{p}$
- $\frac{x}{s} \cdot \frac{y}{t} \in S^{-1}\mathfrak{q} \Rightarrow uxy \in \mathfrak{q} \Rightarrow xy \in \mathfrak{q}$

$\Rightarrow x \in \mathfrak{q}$ or $y \in \sqrt{\mathfrak{q}} = \mathfrak{p}$

$\Rightarrow \frac{x}{s} \in S^{-1}\mathfrak{q}$ or $\frac{y}{t} \in S^{-1}\mathfrak{p}$ $\square$

$\mathfrak{a} \triangleleft A$, S = mult. closed subset.

$$S(\mathfrak{a}) := (S^{-1}\mathfrak{a})^c \triangleleft A.$$

Prop 4.9 $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal. $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$.

Assume $\mathfrak{p}_i \cap S \begin{cases} = \phi & i = 1, \cdots, m \\ \neq \phi & i = m+1, \cdots, n \end{cases}$

Then

$$S^{-1}\mathfrak{a} = \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i \quad \& \quad S(\mathfrak{a}) = \bigcap_{i=1}^{m} \mathfrak{q}_i$$

Pf: $S^{-1}\mathfrak{a} \overset{3.11}{=} \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i \overset{4.8}{=} \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i$

$$S(\mathfrak{a}) = (S^{-1}\mathfrak{a})^c = \bigcap_{i=1}^{m} (S^{-1}\mathfrak{q}_i)^c = \bigcap_{i=1}^{m} \mathfrak{q}_i$$

$$\mathfrak{p}_i \neq \mathfrak{p}_j \Rightarrow S^{-1}\mathfrak{p}_i \neq S^{-1}\mathfrak{p}_j \Rightarrow \text{minimal}$$

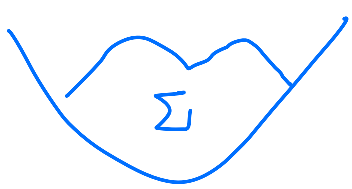
- $\mathfrak{a} = \bigcap \mathfrak{q}_i$ minimal

$\Sigma \subseteq \{ \sqrt{\mathfrak{q}_i} \mid i \}$ is called isolated, if

$\forall \mathfrak{p}' \in \{ \sqrt{\mathfrak{q}_i} \mid i \}, \forall \mathfrak{p} \in \Sigma, \ \mathfrak{p}' \subseteq \mathfrak{p} \Rightarrow \mathfrak{p}' \in \Sigma.$

$$S_\Sigma := A \setminus \bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$$

$$\mathfrak{p} \in \{ \sqrt{\mathfrak{q}_i} \mid i \} \Rightarrow S_\Sigma \cap \mathfrak{p} \begin{cases} = \phi & \mathfrak{p}' \in \Sigma \\ \neq \phi & \mathfrak{p}' \notin \Sigma \end{cases}$$



Thm 4.10 (2^{nd} uniqueness thm) $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal

$\Sigma = \{\mathfrak{p}_{i_1}, \cdots, \mathfrak{p}_{i_m}\}$ = isolated $\Rightarrow$ $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_m}$ is independent of the decomposition.

Pf: $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_m} = S_\Sigma(\mathfrak{a})$ $\square$